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THE ACCELERATED MOTION OF RIGID BODIES IN NON-STEADY STOKES FLOW

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Abstract—The force and torque acting on an accelerating rigid body of arbitrary shape, moving at low Reynolds number through a fluid at rest in infinity, are considered. The expressions found for the force due to pure translation and the torque due to pure rotation of the body each include three tensors which relate the acceleration and velocity to the force and the torque. In the case of combined translation and rotation, three "coupling tensors" are added to each of the above expressions. These expressions are extended for the case of a particle, immersed in a quiet fluid and acted upon by an impulse. Generalized Faxen's theorems are derived for non-steady flows which do not vanish in infinity. Finally, the effect of non-zero initial velocity of the fluid and the body is considered. The stop distance is shown to depend linearly on the initial velocity of the body through a displacement tensor which consists of the traditional quasi-stationary term and an additional tensor. This additional tensor depends on the geometry of the body and on the initial velocity field of the fluid. It is infinite if the kinetic energy of the initial field is infinite. Likewise, the expression for the force acting on the body contains an additional term which depends on time, on the geometry of the problem and on the initial velocity field.

Key Words: Stokes flow, accelerated bodies

1. INTRODUCTION

The motion of rigid bodies, in a viscous fluid, is an important branch of fluid dynamics. To solve the equation of motion of a body one has to determine the fluid-dynamic forces acting on it. These forces are determined from the Navier–Stokes equation with the appropriate initial and boundary conditions:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = \mu \, \Delta \mathbf{u} - \nabla p; \qquad [1]$$

p is the pressure, $\mathbf{u}$ is the velocity of the fluid, ρ its density and μ its viscosity. Let a be a characteristic linear dimension of the rigid body, U a typical magnitude of the fluid's velocity $\mathbf{u}$, τ a typical time scale and ν the kinematic viscosity, [1] may be written in a dimensionless form:

$$\mathrm{Re} \cdot \mathrm{St} \cdot \frac{\partial \mathbf{u}}{\partial t} + \mathrm{Re} \cdot \mathbf{u} \cdot \nabla \mathbf{u} = \Delta \mathbf{u} - \nabla p; \qquad [2]$$

Re is the Reynolds number and St is the Strouhal number,

$$\mathrm{Re} = \frac{Ua}{\nu}, \quad \mathrm{St} = \frac{a}{U\tau}.$$

In the paper we consider only low Re. In most applications one assumes that the motion is steady. Thus, the l.h.s. of [2] is zero and one is left with the steady Stokes equation. From this linear equation one may deduce that every rigid body possesses three resistance tensors which relate the translational and angular velocities to the force and torque acting on the body by the fluid (Happel & Brenner 1965). These tensors are an intrinsic property of the body and depend on its geometry and on the location of the origin of the coordinate system. Their evaluation in general is complicated but for a few symmetric bodies, and bodies which can be approximated by symmetric ones, their analytic form is known.

If the motion is not steady, Re is small and St is large such that $\mathrm{Re} \cdot \mathrm{St} = O(1)$, we obtain, in dimensional form, the time-dependent Stokes equation:

$$\rho \frac{\partial \mathbf{u}}{\partial t} = \mu \, \Delta \mathbf{u} - \nabla p. \qquad [3]$$

One case, for which this condition is satisfied, is the case of a particle, oscillating with an amplitude l, such that $a \gg l$ (Landau & Lifshitz 1959). Another case is that of a translating particle. If the time scale τ is taken to be the Stokes relaxation time, then $\mathrm{Re} \cdot \mathrm{St} \simeq \rho / \rho_p$, where ρ_p is the density of the particle. The condition $\mathrm{St} \gg 1$ leads to the condition $Ua\rho_p/\mu \ll 1$. There exist only a few solutions for the force acting on accelerating bodies. The best known is the Basset (1888) solution for the sphere:

$$\mathbf{F}(t) = -\frac{2\pi a^3}{3}\rho\dot{\mathbf{U}}(t) - 6\pi a\mu\mathbf{U}(t) - 6\pi a^2\sqrt{\pi\mu\rho}\int_0^t \frac{\dot{\mathbf{U}}(\tau)}{\sqrt{t-\tau}}\,d\tau. \qquad [4]$$

The first term on the r.h.s. is the "added mass" which accounts for the changes in the kinetic energy of the fluid. It is the force that would act if the flow were potential. The second term is the steady Stokes resistance and the third term is the Basset memory term; it sums up the effect of the disturbances in the flow caused by the acceleration of the sphere. Other known solutions are the solution for the torque on a rotating sphere (Feuillebois & Lasek 1978), the force on a slightly deformed translating sphere in axisymmetric motion (Lawrence & Weinbaum 1986), the force on a spheroid in axisymmetric motion (Lawrence & Weinbaum 1988) and the torque on a rotating ellipsoid of revolution (Hocquart 1976). The last is given in its Laplace transformed form only. To the best of our knowledge no general theory exists.

Boggio (1907a, b) solved the equation of motion of a sphere with the fluid-dynamic force [4] and an external force. His solution reveals that if ρ/ρ_p is close to one, the character of the particle velocity, is considerably different from that calculated under the quasi-stationary assumption. Arminski & Weinbaum (1979) also solved the equation of motion of a sphere. They considered the motion of a sphere, which started to move from rest under the action of an external force which later ceased to act. They showed that the Basset term does not contribute to the total displacement but, again, the form of the velocity may be very different from the quasi-stationary velocity. Some results of the theory of Brownian motion, based on the time-dependent equation of motion, also reveal a behaviour different from what is expected from the steady Stokes equation. If one adopts the quasi-stationary approach, the velocity autocorrelation function of a Brownian particle, $\phi(t) = \langle \mathbf{U}(t)\mathbf{U}(0)\rangle$, exhibits an exponential decay. In numerical solutions of the full Navier–Stokes equation for spheres, Alder & Wainwright (1970) found that $\phi(t)$ decayed as $t^{-3/2}$ for large t. Such a decay was found by Widom (1971) and Zwanzig & Bixon (1970) in their analysis of the linear equation. It thus seems to use that a general analysis of the unsteady motion of bodies of arbitrary shape would be of interest.

In view of the solution for the sphere [4], we show in the paper that a similar structure appears in the general case, namely an "added mass" tensor, the stationary resistance tensor and a "Basset" tensor. The "Basset" tensor will be defined in terms of solutions of the non-stationary Stokes equation [2]. These solutions, which we name "basic solutions", do not depend on the specific motion of the body.

A closely related problem is that of a body immersed in a flow which is not at rest in infinity. The expression for the force exerted by the fluid on a spherical body was found by Mazur & Bedeaux (1974). Their result, which is a generalization of Faxen's theorem for the non-stationary case, is generalized by us to bodies of arbitrary shape and an analogous relation is derived for the torque.

The stop distance of a particle plays an important role in the process of aerosol sampling and deposition from the atmosphere. A good account of the problem can be found in Fuchs (1964). Suppose a particle, immersed in a fluid is acted upon by some force and attains a velocity $\mathbf{U}(t)$. If at a certain moment $t = 0$ the force ceases to act, the distance, travelled by the particle before it comes to rest, is called the stop distance. In the classical quasi-stationary treatment one assumes that the viscous force is the steady force and the equation of motion is

$$m\dot{\mathbf{U}}(t) = \mathbb{K}\cdot\mathbf{U}(T), \quad \mathbf{U}(0) = \mathbf{U}^0; \qquad [5]$$

$\mathbb{K}$ is the translation tensor defined in Happel & Brenner (1965) and m is the mass of the particle. Thus, for a sphere of radius a and a fluid of viscosity μ, the stop distance calculated by integrating

$\mathbf{U}(t)$ from zero to infinity is $\mathbf{U}^0 m/6\pi a\mu$. In the general asymmetric case, where no coupling with rotation is involved, the terminal displacement vector $\mathbf{X}^\infty$ is

$$\mathbf{X}^\infty = -m\mathbb{K}^{-1}\cdot\mathbf{U}^0. \quad [6]$$

Likewise, the total rotation is

$$\boldsymbol{\phi}^\infty = -\mathbb{\Omega}^{-1}\cdot\mathbb{I}_r\cdot\boldsymbol{\omega}^0, \quad [7]$$

$\mathbb{\Omega}$ is the rotation tensor and $\mathbb{I}_r$ is the moment of inertia tensor.

The basic shortcomings of the quasi-stationary approach are the assumption that the fluid velocity is instantaneously adjusted to the steady Stokes velocity field corresponding to the time-dependent particle velocity and the neglect of the change in the kinetic energy of the fluid. Therefore, a calculation which takes these two points into account will yield a larger stop distance; this was observed by Fuchs (1964). The slowing down of the particle must be accompanied by the slowing down of the fluid. It is therefore reasonable to expect a dependence of the stop distance, and of the force acting on the particle, on the initial velocity field at the time the force ceased to act. This velocity field is not, in general, the steady Stokes field corresponding to the particle velocity. We show that besides the quasi-stationary term the stop distance depends on an additional "inertia" tensor which gives a measure of the energy of the initial flow field to be dissipated by the viscosity. The effect of the initial field is also incorporated into the previously derived expression of the force.

2. A RECIPROCAL FORMULA AND THE CONTINUITY OF THE FLOW

As in steady Stokes flows, the use of a reciprocal formula proves to be useful. A reciprocal formula for unsteady Stokes flows was obtained by Maxey & Riley (1983). Let $\mathbf{u}$ and $\mathbf{v}$ be two Stokes flows in a domain Ω with a boundary Σ and let $\sigma_{ij}[\mathbf{u}]$ be the stress tensor, then:

$$\rho\int_\Omega [u_i(0)v_i(t) - u_i(t)v_i(0)]\,dx = \int_0^t\int_\Sigma \{u_i(t-\tau)\sigma_{ij}[\mathbf{v}(\tau)] - v_i(\tau)\sigma_{ij}[\mathbf{u}(t-\tau)]\}n_j\,dx\,d\tau. \quad [8]$$

In the next section we will use the expression that is obtained from [8] by differentiating it with respect to the time variable t, namely:

$$\rho\int_\Omega [u_i(0)\dot{v}_i(t) - \dot{u}_i(t)v_i(0)]\,dx = \int_\Sigma \{u_i(0)\sigma_{ij}[\mathbf{v}(t)] - v_i(t)\sigma_{ij}[\mathbf{u}(0)]\}n_j\,dx$$
$$+\int_0^t\int_\Sigma \{\dot{u}_i(t-\tau)\sigma_{ij}[\mathbf{v}(\tau)] - v_i(\tau)\dot{\sigma}_{ij}[\mathbf{u}(t-\tau)]\}n_j\,dx\,d\tau. \quad [9]$$

Formulae [8] and [9] are valid only if the flows $\mathbf{u}$ and $\mathbf{v}$ are continuous with respect to t in every internal point of Ω. Their application requires, therefore, a careful choice of the initial and boundary conditions.

A body which starts to move from rest, under the action of an impulse, generates a flow field which is potential at the initial moment $t = +0$ (Batchelor 1967, p. 471), but satisfies the no-slip boundary conditions for $t > 0$. Thus, a discontinuity in the tangential components of the flow at the boundary is permitted but the normal components must be continuous. This assertion is valid for all unsteady Stokes flows, not necessarily generated by the motion of a rigid body. It should be taken into account when [8] and [9] are used.

3. REPRESENTATION OF THE FORCE AND TORQUE

In this section we show that the force and torque, acting on a body of arbitrary shape, are related to the translational and rotational velocities of the body through the resistance tensors. These tensors are expressed in terms of the "basic solutions" which depend on the geometry of the body only and, thus, have to be solved only once for any given shape. The translation and rotation tensors are shown to be symmetric. For non-symmetric bodies rotation and translation are coupled and "coupling" tensors, which are in general not symmetric, have to be added.

A. Force in Translation

Let $\mathbf{u}(x, t)$ be the flow resulting from a translation $\mathbf{U}(t)$ of the body. $\mathbf{u}$ Satisfies [2] with the following initial and boundary conditions:

$$u_i|_\Sigma = U_i(t), \quad u_i|_\infty = 0, \quad u_i|_{t=0} = 0, \quad U_i(0) = 0. \qquad [10]$$

We define the "basic solution" $\mathbf{V}^k$; $\mathbf{V}^k$ satisfies Stokes equation [3] and the following boundary and initial conditions:

$$\mathbf{V}^k|_\Sigma = \mathbf{e}^k, \quad \mathbf{V}^k|_\infty = 0, \quad t > 0 \qquad [11a]$$

and

$$V_i^k|_{t=0} = \frac{\partial \phi^k}{\partial x_i}; \qquad [11b]$$

where ϕ^k is the solution of the following Neumann problem:

$$\Delta\phi^k = 0, \quad \frac{\partial \phi^k}{\partial x_i} n_i|_\Sigma = e_i^k n_i = n_k, \quad \phi^k|_\infty = 0. \qquad [11c]$$

$\mathbf{e}^k$ is the unit vector in the kth direction and $e_i^k = \delta_{ik}$. Due to [11a] and [11c] the normal component of $\mathbf{V}^k$ on the boundary is continuous at $t = 0$ and thus $\mathbf{V}^k$ satisfies the conditions for its continuity at $t = 0$, as posed in section 2. Therefore, we may substitute $\mathbf{u}$ and $\mathbf{V}^k$ in [9] and get:

$$-\rho \int_\Omega \frac{\partial \phi^k}{\partial x_i} \dot{u}_i(t)\, \mathrm{d}x = \int_0^t \mathrm{d}\tau \int_\Sigma \dot{U}_i(t-\tau)\sigma_{ij}[\mathbf{V}^k(\tau)]n_j\, \mathrm{d}x - \int_0^t \mathrm{d}\tau \int_\Sigma e_i^k \dot{\sigma}_{ij}[\mathbf{u}(t-\tau)]n_j\, \mathrm{d}x. \qquad [12]$$

The force $\mathbf{F}$ acting on the body is

$$F_i(t) = \int_\Sigma \sigma_{ij}[\mathbf{u}(t)]n_j\, \mathrm{d}x.$$

Performing the time integration in the last term on the r.h.s. of [12] and transforming the l.h.s. into a surface integral, we obtain

$$F_k(t) = Q_{ki}\dot{U}_i(t) + \int_0^t A_{ki}(t-\tau)\dot{U}_i(\tau)\, \mathrm{d}\tau; \qquad [13]$$

where

$$A_{ki}(t) = \int_\Sigma \sigma_{ij}[\mathbf{V}^k(t)]n_j\, \mathrm{d}x, \quad Q_{ki} = \rho \int_\Sigma \phi^k n_i\, \mathrm{d}x. \qquad [14]$$

The tensor Q_{ki} is the "added mass" (Batchelor 1967) which, in the case of a translating sphere, has the form $-\frac{2}{3}\rho\pi a^3\delta_{ki}$. The term $Q_{ki}\dot{U}_i(t)$ is the force that would act on the body if the flow caused by the motion of the body were potential. $A_{ki}(t)$ is the ith component of the force that the flow $\mathbf{V}^k$ applies on the body. The tensor $\mathbb{A}$ depends on the basic solutions $\mathbf{V}^k$ which, themselves, depend on the geometry of the body, time and the density and viscosity of the fluid.

Since the boundary conditions [11a–c], for $\mathbf{V}^k$, do not depend on time it seems reasonable to expect $\mathbf{V}^k$ to converge to the steady solution as time tends to infinity. It was shown by Hocquart & Hinch (1983) that the flow due to an impulsive force decays as $t^{-3/2}$ and the flow due to an impulsive couple, applied to centrally symmetric bodies, decays as $t^{-5/2}$. The rates of convergence of the basic solutions are therefore expected to be $t^{-1/2}$ and $t^{-3/2}$, respectively. The first is the known rate for spheres and for spheroids, translating parallel to their axis of symmetry (Lawrence & Weinbaum 1986, 1988); the second is the known rate for rotating spheres (Feuillebois & Lasek 1978). Moreover, Heywood (1974) showed that the solution $\mathbf{u}$ of the exterior non-steady Stokes problem with time-independent boundary conditions converges to the steady solution $\bar{\mathbf{u}}$ with the same boundary conditions, in every bounded subset Ω' of Ω in the norm of $L_2(\Omega')$. His proof shows also that the time derivatives converge globally to zero in $L_2(\Omega)$:

$$\sum_{i=1}^3 \int_{\Omega'} (u_i - \bar{u}_i)^2\, \mathrm{d}x \xrightarrow[t\to\infty]{} 0, \quad \sum_{i=1}^3 \int_\Omega \left[\frac{\partial}{\partial t} u_i(x,t)\right]^2 \mathrm{d}x \leqslant \frac{K}{t}. \qquad [15]$$

(The proof, though, is restricted to cases with no jump in the boundary.) The flow $\mathbf{V}^k$ may, therefore, be expressed as the sum of the steady flow $\bar{\mathbf{V}}^k$ and a non-steady, asymptotically vanishing flow, $\hat{\mathbf{V}}^k$;

$$\mathbf{V}^k = \bar{\mathbf{V}}^k + \hat{\mathbf{V}}^k. \tag{16}$$

$\bar{\mathbf{V}}^k$ satisfies the steady Stokes equation and the boundary conditions:

$$\bar{V}_i^k|_\Sigma = \delta_{ik}, \quad \bar{V}_i^k|_\infty = 0. \tag{17}$$

$\hat{\mathbf{V}}^k$ satisfies the non-steady Stokes equation with the boundary and initial conditions:

$$\hat{V}_i^k|_\Sigma = 0, \quad \hat{V}_i^k|_\infty = 0, \quad \hat{V}_i^k|_{t=0} = \frac{\partial \phi^k}{\partial x_i} - \bar{V}_i^k. \tag{18}$$

$\hat{\mathbf{V}}^k$ converges asymptotically to zero,

$$\hat{\mathbf{V}}^k \xrightarrow[t\to\infty]{} 0.$$

The tensor $\mathbb{A}(t)$ may therefore be decomposed into the steady translation tensor $\mathbb{K}$,

$$K_{ki} = \int_\Sigma \sigma_{ij}[\bar{\mathbf{V}}^k] n_j \,\mathrm{d}x, \tag{19}$$

and the Basset tensor,

$$B_{ki}(t) = \int_\Sigma \sigma_{ij}[\hat{\mathbf{V}}^k(t)] n_j \,\mathrm{d}x, \tag{20}$$

i.e.

$$\mathbb{A}(t) = \mathbb{K} + \mathbb{B}(t) \tag{21}$$

such that

$$\mathbb{B}(t) \xrightarrow[t\to\infty]{} 0.$$

The expression for the force [13] assumes now the form

$$F_k(t) = Q_{ki}\dot{U}_i(t) + K_{ki}U_i(t) + \int_0^t B_{ki}(t-\tau)\dot{U}_i(\tau)\,\mathrm{d}\tau. \tag{22}$$

Equation [22] has been derived under the assumption that the initial velocity of the body and the fluid were zero. However application of [9] shows that if the initial velocity of the body was not zero and if the initial fluid velocity was the steady Stokes flow, corresponding to the initial velocity of the body, then [22] is still valid.

B. The Symmetry of the Resistance Tensors

We now show that the tensor $\mathbb{A}$ is symmetric. This symmetry implies the symmetry of the Basset tensor $\mathbb{B}$. We first note that the added mass tensor $\mathbb{Q}$ is symmetric (Batchelor 1967). This is a consequence of the following reciprocal identity for the two solutions ϕ^i and ϕ^k of the Laplace equation:

$$\int_\Sigma \left(\phi^k \frac{\partial \phi^i}{\partial x_j} - \phi^i \frac{\partial \phi^k}{\partial x_j}\right) n_j \,\mathrm{d}x = 0. \tag{23}$$

Now substituting the solutions $\mathbf{V}^k$ and $\mathbf{V}^i$ of [11a–c] in the reciprocal formula [8] we obtain

$$\int_\Omega \left[\frac{\partial \phi^i}{\partial x_j} V_j^k(t) - \frac{\partial \phi^k}{\partial x_j} V_j^i(k)\right] \mathrm{d}x = \int_0^t \mathrm{d}\tau \int_\Sigma \{\delta_{ij}\sigma_{jl}[\mathbf{V}^k(\tau)] - \delta_{kj}\sigma_{jl}[\mathbf{V}^i(t-\tau)]\} n_l \,\mathrm{d}x. \tag{24}$$

Transforming the l.h.s. into a surface integral on the body's surface it reduces to [23] and thus vanishes; [24] then becomes

$$\int_0^t [A_{ki}(\tau) - A_{ik}(\tau)]\,d\tau = 0.$$

Since this identity holds for every t, $\mathbb{A}$ must be symmetric.

C. Torque in Rotation

The treatment of the rotational case runs along exactly the same lines as that of the translational case and differs only in the choice of the basic solutions. Consider a body starting to rotate from rest at an angular velocity $\boldsymbol{\omega}(t)$ about the origin of the coordinate system. Let $\mathbf{u}$ be the Stokes flow field resulting from this rotation; $\mathbf{u}$ satisfies the following initial and boundary conditions:

$$u_i|_{t=0} = 0; \quad u_i|_\Sigma = \varepsilon_{ijk}\omega_j(t)x_k; \quad \omega_j(0) = 0. \tag{25}$$

We choose the basic solution in the form:

$$\mathbf{W}^k = \bar{\mathbf{W}}^k + \hat{\mathbf{W}}^k.$$

$\bar{\mathbf{W}}^k$ is the steady Stokes solution for the following boundary conditions:

$$\bar{W}_i^k|_\Sigma = \varepsilon_{ijl}\delta_{kj}x_l = \varepsilon_{ikl}x_l, \quad \bar{W}_i^k|_\infty = 0. \tag{26a}$$

$\hat{\mathbf{W}}^k$ is the non-steady Stokes solution corresponding to the following conditions:

$$\hat{W}_i^k|_\Sigma = 0, \quad \hat{W}_i^k|_\infty = 0, \quad \hat{W}_i^k|_{t=0} = \frac{\partial\psi^k}{\partial x_i} - \bar{W}_i^k. \tag{26b}$$

ψ^k again is the solution of the outer Neumann problem of the Laplace equation:

$$\Delta\psi^k = 0, \quad \frac{\partial\psi^k}{\partial x_i}n_i|_\Sigma = \varepsilon_{ikj}x_j n_i, \quad \psi^k|_\infty = 0. \tag{26c}$$

With this definition the continuity condition at $t = 0$ is satisfied. The torque $\mathbf{M}$ is given in terms of the steady rotation tensor Ω,

$$\Omega_{ki} = \int_\Sigma \varepsilon_{ilj}x_l\sigma_{jm}[\bar{\mathbf{W}}^k]n_m\,dx, \tag{27}$$

the Basset tensor,

$$b_{ki}(t) = \int_\Sigma \varepsilon_{ilj}x_l\sigma_{jm}[\hat{\mathbf{W}}^k(t)]n_m\,dx, \tag{28}$$

and the potential tensor,

$$H_{ki} = \rho\int_\Sigma \varepsilon_{ilj}\psi^k x_l n_j\,dx: \tag{29}$$

$$M_k(t) = H_{ki}\dot{\omega}_i(t) + \Omega_{ki}\omega_i(t) + \int_0^t b_{ki}(t-\tau)\dot{\omega}_i(\tau)(\tau)\,d\tau. \tag{30}$$

The symmetry of these tensors is proved in the same way as that of the case of translation. The case of rotating bodies was solved by Feuillebous & Lasek (1978) for a sphere and by Hocquart (1976) for an ellipsoid of revolution rotating about its symmetry axis, the last solution is given only in its Laplace transform form.

D. General Rigid Body Motion

As in steady motions, skew bodies in non-steady motions are acted upon by a torque in translation and a force in rotation. These are related to each other by the "potential", "steady" and "Basset" coupling tensors. Substitution of the flow due to translation in the reciprocal formula together with the basic solution of rotation on the one hand and the flow due to rotation with the basic solution of translation on the other, yields the expressions for the coupling tensors. The potential coupling tensor $\mathbb{G}$ has the form

$$G_{ki} = \int_{\Sigma} \varepsilon_{jil} \phi^k x_l n_j \, dx = \int_{\Sigma} \psi^i n_k \, dx. \qquad [31]$$

The Basset tensor is

$$\beta_{ki}(t) = \int_{\Sigma} \varepsilon_{jil} x_l \sigma_{jm} [\hat{\mathbf{V}}^k(t)] n_m \, dx = \int_{\Sigma} \sigma_{kj} [\hat{\mathbf{W}}^i(t)] n_j \, dx. \qquad [32]$$

The steady tensor $\mathbb{C}$ has the same form except that the steady solutions $\bar{\mathbf{V}}^k$ and $\bar{\mathbf{W}}^i$ replace $\hat{\mathbf{V}}^k$ and $\hat{\mathbf{W}}^i$. These tensors are in general not symmetric. The motion of a body may be described in a six-dimensional generalized space of translation and rotation. We denote the generalized force and velocity $\mathscr{F}$ and $\mathscr{U}$ by

$$\mathscr{F} = \begin{bmatrix} \mathbf{F} \\ \mathbf{M} \end{bmatrix}, \quad \mathscr{U} = \begin{bmatrix} \mathbf{U} \\ \boldsymbol{\omega} \end{bmatrix}; \qquad [33]$$

and the potential, steady and Basset resistance tensors $\mathbb{P}$, $\mathbb{R}$ and $\mathbb{T}$ by

$$\mathbb{P} = -\begin{bmatrix} \mathbb{Q} & \mathbb{G} \\ \mathbb{G}^t & \mathbb{H} \end{bmatrix}, \quad \mathbb{R} = -\begin{bmatrix} \mathbb{K} & \mathbb{C} \\ \mathbb{C}^t & \mathbb{\Omega} \end{bmatrix}, \quad \mathbb{T} = -\begin{bmatrix} \mathbb{B} & \boldsymbol{\beta} \\ \boldsymbol{\beta}^t & \mathbb{b} \end{bmatrix}. \qquad [34]$$

The expression for the generalized force is

$$\mathscr{F} = -\mathbb{P} \cdot \dot{\mathscr{U}} - \mathbb{R} \cdot \mathscr{U} - \int_0^t \mathbb{T}(t-\tau) \cdot \dot{\mathscr{U}}(\tau) \, d\tau. \qquad [35]$$

E. A Remark on Applicability

The practical use of [35] depends on finding the basic solutions. Except for some symmetric cases for which the analytical form of the resistance tensors is known, one would have to solve them numerically. The boundary elements method seems to be most suitable for the solution of the potential and the steady tensors. There exists quite an extensive literature on the topic, of which we mention only Hess (1975), Youngren & Acrivos (1975) and Brebbia *et al.* (1987). The solution of the Basset tensors is more complicated as it is a non-steady problem; so far no related publications are known to us.

4. THE MOTION DUE TO AN IMPULSE

In this section we derive expressions for the force acting on a body brought instantaneously into motion by an impulsive force. Let $\delta > 0$ be arbitrarily small and u_i be the flow causes by the motion of the particle $U_i(t)$:

$$U_i(t) = H(t) U_i^0 + f_i(t). \qquad [36]$$

$H(t)$ is the unit step function at $t = 0$, U_i^0 is the jump and $f_i(t)$ is continuous for $t \geqslant 0$, $f_i(0) = 0$. Since u_i is continuous for every $t > 0$, then

$$\rho \int_0^{t-\delta} \frac{\partial}{\partial \tau} \int_{\Omega} u_i(t-\tau) V_i^k(\tau) \, dx \, d\tau = \rho \int_{\Omega} [u_i(\delta) V_i^k(t-\delta) - u_i(t) V_i^k(0)] \, dx. \qquad [37]$$

Differentiating this expression with respect to t, following the same procedure as in section 3 and bearing in mind that $V_i^k|_{\Sigma} = \delta_{ik}$, [12] now assumes the form

$$\rho \int_{\Omega} \left[u_i(\delta) \dot{V}_i^k(t-\delta) - \frac{\partial \phi^k}{\partial x_i} \dot{u}_i(t) \right] dx = -F_k(t) + \int_{\Sigma} U_i(\delta) \sigma_{ij} [V^k(t-\delta)] n_j \, dx$$
$$+ \int_0^{t-\delta} d\tau \int_{\Sigma} \dot{U}(t-\tau) \sigma_{ij} [V^k(\tau)] n_j \, dx. \qquad [38]$$

(F_k is the force acting on the body.) Letting $\delta \to 0$, the boundary velocity $U_i(\delta) \to U_1^0$. Since, according to Batchelor (1967), the limit of $u_i(\delta)$ is a potential flow, we may write

$$u_i(x, \delta) \xrightarrow[\delta \to 0]{} \frac{\partial}{x_i} G(x) \tag{39}$$

where

$$G(x) = \int_\Sigma \frac{1}{|x - y|} f(y)\,\mathrm{d}y \tag{40}$$

for some density function f. The left term on the l.h.s. of [38] vanishes due to the zero divergence of $\dot{V}_i^k$ and its vanishing on Σ. the surface integral on the r.h.s. of [38] becomes

$$U_i^0 \int_\Sigma \sigma_{ij}[V^k(t)] n_j \,\mathrm{d}x$$

and the expression for the force is

$$F_k(t) = Q_{ki}\dot{U}_i(t) + A_{ki}(t) U_i^0 + \int_0^t A_{ki}(t - \tau)\dot{U}_i(\tau)\,\mathrm{d}\tau, \tag{41}$$

where $\mathbb{A}$ and $\mathbb{Q}$ are as in [14]. Decomposing $\mathbb{A}$ into $\mathbb{K} + \mathbb{B}(t)$, as in [21], we finally obtain:

$$F_k(t) = Q_{ki}\dot{U}_i(t) + K_{ki} U_i(t) + B_{ki}(t) U_i^0 + \int_0^t B_{ki}(t - \tau)\dot{U}_i(\tau)\,\mathrm{d}\tau. \tag{42}$$

Expression [42] for the force in the presence of an initial jump differs from [22] for the continuous case in the inclusion of the additional term $B_{ki}(t)U_i^0$. It could have been derived formally by taking the time derivative of [36], with $\dot{H}(t) = \delta(t)$, and substituting it in [22].

5. ARBITRARY STOKES FLOW—FAXEN'S THEOREMS

In this section we derive generalized Faxen's theorems for the force and torque acting on a body immersed in a Stokes flow which does not vanish in infinity. The method of derivation is similar to the method employed by Kim & Miflin (1985) and by Durlofsky *et al.* (1987) to obtain the hydrodynamic interaction of several bodies in steady Stokes flows. Let $\mathbf{u}$ be the undisturbed Stokes flow defined in the whole space, not vanishing in infinity and being zero at $t = 0$:

$$\rho \frac{\partial}{\partial t} u_i = \mu\, \Delta u_i - \frac{\partial p}{\partial x_i}, \quad \frac{\partial u_i}{\partial x_i} = 0; \quad u_i\big|_{t=0} = 0. \tag{43}$$

Let $\mathbf{U}$ be the rigid motion of the body consisting of translation v_i and rotation ω_i:

$$U_i(t) = v_i(t) + \varepsilon_{ijk}\omega_j(t) x_k; \quad U_i(0) = 0.$$

Let $\boldsymbol{\Omega}$ be the part of space exterior to the body, $\boldsymbol{\Omega}_i$ the part occupied by the body and Σ the surface of the body. We define in $\boldsymbol{\Omega}$ the following Stokes flow $\mathbf{w}$:

$$w_i\big|_\Sigma = U_i - u_i\big|_\Sigma, \quad w_i\big|_\infty = 0, \quad w_i\big|_{t=0} = 0. \tag{44}$$

Because of the linearity and uniqueness of the solution of the Stokes equation, the disturbed flow is $\mathbf{u} + \mathbf{w}$ and the force acting on the body arises from both flows. Let $\mathbf{F}^1$ be the contribution of $\mathbf{u}$ to the force:

$$F_k^1 = \int_\Sigma \sigma_{kj}[\mathbf{u}] n_j \,\mathrm{d}x. \tag{45}$$

Since $\mathbf{u}$ is defined in the whole space, [45] may be transformed into an integral over the volume $\boldsymbol{\Omega}_i$:

$$F_k^1 = \int_{\Omega_i} \frac{\partial}{\partial x_j}\sigma_{kj}[\mathbf{u}]\,\mathrm{d}x = \rho \int_{\Omega_i} \frac{\partial}{\partial t} u_k \,\mathrm{d}x. \tag{46}$$

F_k^1 may be interpreted as a buoyancy term. To derive the contribution $\mathbf{F}^2$ of $\mathbf{w}$, we substitute $\mathbf{w}$ and the basic solution $\mathbf{V}^k$ of [11a–c] and [16] in the reciprocal formula [9]. Substituting the boundary conditions [44] for $\mathbf{w}$, the overall force acting on the body is

$$F_k(t) = \rho \int_{\Omega_i} \dot{u}_k \,\mathrm{d}x + Q_{ki}\dot{v}_i + G_{ki}\dot{\omega}_i - \rho \int_\Sigma \dot{u}_i \phi^k n_i \,\mathrm{d}x + K_{ki}v_i + C_{ki}\omega_i$$

$$- \int_\Sigma u_i \sigma_{ij}[\mathbf{V}^k] n_j \,\mathrm{d}x + \int_0^t B_{ki}(t-\tau)\dot{v}_i(\tau)\,\mathrm{d}\tau$$

$$+ \int_0^t \beta_{ki}(t-\tau)\dot{\omega}_i(\tau)\,\mathrm{d}\tau - \int_0^t \mathrm{d}\tau \int_\Sigma \sigma_{ij}[\hat{\mathbf{V}}^k(t-\tau)]\dot{u}_i(\tau)\,\mathrm{d}x. \qquad [47]$$

For the case of the sphere this expression reduces to that given by Mazur & Bedeaux (1974). The torque is obtained similarly with the aid of the basic solution of rotation $\mathbf{W}^k$:

$$M_k(t) = \rho\varepsilon_{kij} \int_{\Omega_i} x_i \dot{u}_j \,\mathrm{d}x + H_{ki}\dot{\omega}_i + G_{ik}\dot{v}_i - \rho \int_\Sigma \dot{u}_i \psi^k n_i \,\mathrm{d}x + \Omega_{ki}\omega_i + C_{ik}v_i - \int_\Sigma u_i \sigma_{ij}[\mathbf{W}^k] n_j \,\mathrm{d}x$$

$$+ \int_0^t b_{ki}(t-\tau)\dot{\omega}_i(\tau)\,\mathrm{d}\tau + \int_0^t \beta_{ik}(t-\tau)\dot{v}_i(\tau)\,\mathrm{d}\tau - \int_0^t \mathrm{d}\tau \int_\Sigma \dot{u}_i(\tau)\sigma_{ij}[\hat{\mathbf{W}}^k(t-\tau)]n_j \,\mathrm{d}x. \qquad [48]$$

As an example we derive Faxen's theorem for the torque on a sphere of radius a. The basic solution for rotation is obtained form Feuillebois & Lasek (1978):

$$\psi^k = 0, \quad \sigma_{il}[\mathbf{W}^k]n_l = -\frac{3\mu}{a}\varepsilon_{ikj}x_j, \quad \sigma_{il}[\hat{\mathbf{W}}^k]n_l = -\sqrt{\mu\rho}\frac{1}{\sqrt{t}}\, i\mathrm{erfc}\left[\left(\frac{\nu t}{a^2}\right)^{1/2}\right]\exp\left(\frac{\nu t}{a^2}\right)\varepsilon_{ikj}x_j. \qquad [49]$$

The sphere's motion may be assumed purely rotational since the translation components will cancel in [48], i.e.

$$U_i = \varepsilon_{ikj}\omega_k(t)x_j.$$

Substitution of [49] in [48] yields:

$$M_k(t) = \varepsilon_{kij}\rho \int_{\Omega_i} x_i \dot{u}_j \,\mathrm{d}x - 8\pi\mu a^3 \omega_k(t) + \frac{3\mu}{a}\varepsilon_{kij}\int_\Sigma x_i u_j \,\mathrm{d}x$$

$$-\frac{8\pi a^4}{3}\sqrt{\mu\rho}\int_0^t \dot{\omega}_k(\tau)\frac{1}{\sqrt{t-\tau}}\, i\mathrm{erfc}\left\{\left[\frac{\nu(t-\tau)}{a^2}\right]^{1/2}\right\}\exp\left[\frac{\nu(t-\tau)}{a^2}\right]\mathrm{d}\tau$$

$$+\sqrt{\mu\rho}\,\varepsilon_{kij}\int_0^t \frac{1}{\sqrt{t-\tau}}\, i\mathrm{erfc}\left\{\left[\frac{\nu(t-\tau)}{a^2}\right]\right\}\exp\left[\frac{\nu(-\tau)}{a^2}\right]\int_\Sigma x_i \dot{u}_j(\tau)\,\mathrm{d}x\,\mathrm{d}\tau. \qquad [50]$$

The mean value theorem for surface integrals on a sphere (Courant & Hilbert 1962) states

$$\int_\Sigma f(x)\,\mathrm{d}x = 4\pi a^2 \sum_{n=0}^\infty \frac{a^{2n}}{(2n+1)!}\Delta^n f(0).$$

Applied to [50] the torque may be expressed in the alternative form:

$$M_k(t) = \varepsilon_{kij}\int_{\Omega_i} x_i\dot{u}_j(t)\,\mathrm{d}x - 8\pi\mu a^3\omega_k(t) - \frac{8\pi\mu a^4}{3}\sqrt{\mu\rho}\int_0^t \dot{\omega}_k(\tau)\frac{1}{\sqrt{t-\tau}}\,i\mathrm{erfc}$$

$$\times\left\{\left[\frac{\nu(t-\tau)}{a^2}\right]^{1/2}\right\}\exp\left[\frac{\nu(t-\tau)}{a^2}\right]\mathrm{d}\tau + 12\pi\mu a\sum_{n=1}^{\infty}\frac{a^{2n}}{(2n+1)!}2n\,\Delta^{n-1}\varepsilon_{kij}\frac{\partial}{\partial x_i}u_j(t)\bigg|_{|x|=0}$$

$$+4\pi a^2\sqrt{\mu\rho}\sum_{n=1}^{\infty}\frac{a^{2n}}{(2n+1)!}2n\int_0^t\frac{1}{\sqrt{t-\tau}}\,i\mathrm{erfc}\left\{\left[\frac{\nu(t-\tau)}{a^2}\right]^{1/2}\right\}\exp\left[\frac{\nu(t-\tau)}{a^2}\right]$$

$$\times\Delta^{n-1}\varepsilon_{kij}\frac{\partial}{\partial x_i}\dot{u}_j(\tau)\bigg|_{|x|=0}\mathrm{d}\tau,$$

assuming that this series converges. It should be noted that, unlike steady Stokes flows, $\Delta^n\mathbf{u}$ does not vanish for $n \geqslant 2$.

6. STOP DISTANCE OF PARTICLES

We being our investigation by examining two limiting cases. The first is the case of a particle brought to its settling velocity under the action of some external force, such that fluid velocity is the steady Stokes flow, and then left to deccelerate. The second is the case of a particle set in motion by an impulse and then left to deccelerate.

In the first case [22] is applicable. The equation of motion of the particle is

$$m\dot{\mathbf{U}}(t) = \mathbb{Q}\cdot\dot{\mathbf{U}}(t) + \mathbb{K}\cdot\mathbf{U}(t) + \int_0^t \mathbb{B}(t-\tau)\cdot\dot{\mathbf{U}}(\tau)\,\mathrm{d}\tau, \quad \mathbf{U}(0) = \mathbf{U}^0. \tag{51}$$

Applying the Laplace transform to this equation, the transform of the velocity

$$\tilde{\mathbf{U}}(p) = \mathbb{M}(p)\cdot\mathbf{U}^0, \tag{52}$$

where

$$\mathbb{M}(p) = [p(m\mathbb{I} - \mathbb{Q}) - \mathbb{K} - p\tilde{\mathbb{B}}]^{-1}\cdot[m\mathbb{I} - \mathbb{Q} - \tilde{\mathbb{B}}]. \tag{53}$$

($\mathbb{I}$ is the identity matrix.) The terminal displacement $\mathbf{X}^\infty$ is

$$\mathbf{X}^\infty = \int_0^\infty \mathbf{U}(t)\,\mathrm{d}t = \lim_{p\to 0}\tilde{\mathbf{U}}(p). \tag{54}$$

For translational motions $\mathbb{B}(t)$ is of order $t^{-1/2}$ for large t so that $\tilde{\mathbb{B}}(p)$ is of order $p^{-1/2}$ for small p [see, for example, for spheroids, Lawrence & Weinbaum (1988)]. Thus, $\mathbf{X}^\infty$ is infinite. It is obvious that the rate of decay of $\mathbb{B}(t)$, which reflects the rate of spread of disturbances in the fluid, is the cause to the infinite distance; a faster decay would result in a finite result. Indeed, the equation of angular velocity derived form [30] is

$$\mathbb{I}_r\cdot\dot{\omega}(t) = \mathbb{H}\cdot\dot{\omega}(t) + \Omega\cdot\omega(t) + \int_0^t \mathbb{b}(t-\tau)\cdot\dot{\omega}(\tau)\,\mathrm{d}\tau, \quad \omega(0) = \omega^0. \tag{55}$$

$\mathbb{H}$ and $\mathbb{b}$ are the added moment of inertia and the rotation Basset tensors, respectively. The corresponding matrix $\mathbb{M}(p)$ is

$$\mathbb{M}(p) = [p(\mathbb{I}_r - \mathbb{H}) - \Omega - p\tilde{\mathbb{b}}]^{-1}\cdot[\mathbb{I}_r - \mathbb{H} - \tilde{\mathbb{b}}]. \tag{56}$$

For a sphere of radius a, $\mathbb{H} = 0$ and $\tilde{\mathbb{b}}(p)$ is obtained by Laplace transforming the results of Feuillebois & Lasek (1978):

$$\tilde{\mathbb{b}}(p) = -\frac{8\pi a^5\sqrt{\mu\rho}}{3\left(\sqrt{\dfrac{\mu}{\rho}} + \sqrt{p}\right)}\mathbb{I}. \tag{57}$$

The total rotation $\boldsymbol{\phi}^{\infty}$ is

$$\boldsymbol{\phi}^{\infty} = -\mathbb{\Omega}^{-1} \cdot \left[\frac{8\pi\rho a^5}{3}\mathbb{I} + \mathbb{I}_r\right] \cdot \boldsymbol{\omega}^0 = -6\mathbb{\Omega}^{-1} \cdot \mathbb{I}_r \cdot \boldsymbol{\omega}^0. \quad [58]$$

It is six times larger than the quasi-stationary result.

In the second limiting case a particle is set in motion by an impulse. The equation of motion is derived from [47]:

$$m\dot{\mathbf{U}}(t) = \mathbb{Q} \cdot \dot{\mathbf{U}}(t) + \mathbb{K} \cdot \mathbf{U}(t) + \mathbb{B}(t) \cdot \mathbf{U}^0 + \int_0^t \mathbb{B}(t-\tau) \cdot \dot{\mathbf{U}}(\tau)\,d\tau; \quad \mathbf{U}(0) = \mathbf{U}^0. \quad [59]$$

The matrix $\mathbb{M}(p)$ is now

$$\mathbb{M}(p) = [p(m\mathbb{I} - \mathbb{Q}) - \mathbb{K} - p\tilde{\mathbb{B}}]^{-1} \cdot [m\mathbb{I} - \mathbb{Q}], \quad [60]$$

so that the terminal displacement is finite:

$$\mathbf{X}^{\infty} = -\mathbb{K}^{-1} \cdot [m\mathbb{I} - \mathbb{Q}] \cdot \mathbf{U}^0. \quad [61]$$

This is just the classical quasi-stationary result, which ignores the inertia of the fluid, with the additional contribution of the added mass. In most realistic situations the settling velocity is not reached and the initial fluid velocity is in an intermediate state between the two described extremes.

We now derive the general relation between the initial velocity field and the force acting on the particle. From this relation we obtain the relation for the terminal displacement. For the sake of brevity we first derive the expressions for pure translation and then extend them for a general body motion. Suppose that the particle velocity, at time $t = 0$, was $\mathbf{U}^0$ and the initial velocity of the fluid was $\mathbb{W} \cdot \mathbf{U}^0$. We assume that the tensor field $\mathbb{W}$ satisfies the following:

$$\frac{\partial W_i^k}{\partial x_i} = 0, \quad \mathbf{W}^k|_{\Sigma} = \mathbf{e}^k, \quad \mathbf{W}^k|_{\infty} = 0. \quad [62]$$

$\mathbf{e}^k$ is the unit vector in the kth direction.

The substitution in [9] of the Stokes flow field defined by the initial condition [62] and the basic solution of translation $\mathbb{V}$, defined by [11a–c], yields, after some manipulation, the expression for the force acting on the particle:

$$\mathbf{F}(t) = \mathbb{Q} \cdot \dot{\mathbf{U}}(t) + \mathbb{K} \cdot \mathbf{U}(t) + \int_0^t \mathbb{B}(t-\tau) \cdot \dot{\mathbf{U}}(\tau)\,d\tau + \mathbb{B}(t) \cdot \mathbf{U}^0 - \rho\left[\int_{\Omega} \dot{\mathbb{V}}^t \cdot \mathbb{W}\,dx\right] \cdot \mathbf{U}^0. \quad [63]$$

(The tensor $\mathbb{V}^t$ is the transpose of $\mathbb{V}$.) This last expression is a more general form of the force and includes [22] as a special case. The last term on the r.h.s. is the explicit relation between the force and the initial field. Applying the Laplace transform $\mathscr{L}$ to the equation of motion arising from [63] one obtains:

$$[p(m\mathbb{I} - \mathbb{Q}) - \mathbb{K} - p\tilde{\mathbb{B}}] \cdot \tilde{\mathbf{U}} = [m\mathbb{I} - \mathbb{Q}] \cdot \mathbf{U}^0 - \mathscr{L}\left[\rho \int_{\Omega} \dot{\mathbb{V}}^t \cdot \mathbb{W}\,dx\right] \cdot \mathbf{U}^0. \quad [64]$$

Recalling the decomposition [16] of the basic velocity:

$$\mathbb{V} = \bar{\mathbb{V}} + \hat{\mathbb{V}}, \quad \hat{V}_i^k|_{t=0} = \frac{\partial \phi^k}{\partial x_i} - \bar{V}_i^k,$$

we get

$$\dot{\mathbb{V}} = \dot{\hat{\mathbb{V}}}.$$

Passing to the limit $p \to 0$ in the last term on the r.h.s. of [64]:

$$\lim_{p\to 0} \mathscr{L}\left[\rho \int_{\Omega} \dot{\mathbb{V}}^t \cdot \mathbb{W}\,dx\right] = -\mathbb{Q} + \rho \int_{\Omega} \bar{\mathbb{V}}^t \cdot \mathbb{W}\,dx + \lim_{t\to\infty} \rho \int_{\Omega} \hat{\mathbb{V}}^t(t) \cdot \mathbb{W}\,dx. \quad [65]$$

Thus, the expression for the terminal displacement is

$$\mathbf{X}^0 = -m\mathbb{K}^{-1}\cdot\mathbf{U}^0 - \rho\,\mathbb{K}^{-1}\cdot\left[\int_\Omega \mathbb{V}^t\cdot\mathbb{W}\,\mathrm{d}x + \lim_{t\to\infty}\int_\Omega \hat{\mathbb{V}}^t\cdot\mathbb{W}\,\mathrm{d}x\right]\cdot\mathbf{U}^0. \qquad [66]$$

The conditions for the convergence of the last integrals are different for translation and for axisymmetric rotation. For translation $\mathbb{V}$ and $\hat{\mathbb{V}}$ decrease in infinity as $1/|x|$ so that $\mathbb{W}$ must decrease faster than $1/|x|^2$. In axisymmetric rotation, the corresponding terms of $\mathbb{V}$ and $\hat{\mathbb{V}}$ decrease as $1/|x|^2$. In general, in order for the last term on the r.h.s. of [66] to vanish, it is sufficient that $\mathbb{W}$ decreases faster than $1/|x|^2$. In this case:

$$\mathbf{X}^\infty = -m\mathbb{K}^{-1}\cdot\mathbf{U}^0 - \rho\,\mathbb{K}^{-1}\cdot\left[\int_\Omega \mathbb{V}^t\cdot\mathbb{W}\,\mathrm{d}x\right]\cdot\mathbf{U}^0. \qquad [67]$$

Expressions [66] and [67] include the two extreme cases, described in the beginning of this section, as special cases. Indeed, if the initial velocity $\mathbb{W}$ is the steady stokes field $\mathbb{V}$, then the volume integrals become infinite. If $\mathbb{W}$ is a potential flow, then the volume integrals reduce to surface integrals and [61] is recovered.

It is possible to generalize [66] and [67] to the six-dimensional space of translation and rotation. Let $\boldsymbol{\xi}^\infty$ be the terminal dispalcement vector and $\mathcal{U}$ the body's velocity:

$$\boldsymbol{\xi}^\infty = \begin{bmatrix}\mathbf{X}^\infty\\ \boldsymbol{\phi}^\infty\end{bmatrix}\quad \mathcal{U} = \begin{bmatrix}\mathbf{U}\\ \boldsymbol{\omega}\end{bmatrix}. \qquad [68]$$

The initial velocity field $\mathbb{W}$ is defined as the 3×6 matrix:

$$\mathbb{W} = ({}^{\mathrm{T}}\mathbb{W}, {}^{\mathrm{R}}\mathbb{W}); \qquad [69]$$

where ${}^{\mathrm{T}}\mathbb{W}$ is the initial translation field defined by [62] and ${}^{\mathrm{R}}\mathbb{W}$ is the initial rotation field defined correspondingly. The steady Stokes field $\mathbb{V}$ is a 3×6 matrix field composed of the steady solutions of translation ${}^{\mathrm{T}}\mathbb{V}$ and of rotation ${}^{\mathrm{R}}\mathbb{V}$ as defined in [16] and [26]a–c] in section 3:

$$\mathbb{V} = ({}^{\mathrm{T}}\mathbb{V}, {}^{\mathrm{R}}\mathbb{V}). \qquad [70]$$

The resistance tensor matrix $\mathbb{R}$ is composed of the translation tensor $\mathbb{K}$, the rotation tensor Ω and the coupling tensor $\mathbb{C}$:

$$\mathbb{R} = -\begin{bmatrix}\mathbb{K} & \mathbb{C}\\ \mathbb{C}^t & \Omega\end{bmatrix}. \qquad [71]$$

The particle's generalized inertia tensor matrix $\mathbb{I}_p$ is composed of the mass of the particle m and its moment of inertia tensor $\mathbb{I}_r$:

$$\mathbb{I}_p = \begin{bmatrix} m\mathbb{I} & 0\\ 0 & \mathbb{I}_r\end{bmatrix}. \qquad [72]$$

In this designation the translation–rotation terminal displacement, subjected to the assumption that the last term on the r.h.s. of [66] vanishes is

$$\boldsymbol{\xi}^\infty = \mathbb{D}\cdot\mathcal{U}^0, \qquad [73]$$

where the displacement matrix $\mathbb{D}$ is

$$\mathbb{D} = \mathbb{R}^{-1}\cdot\left[\mathbb{I}_p + \rho\int_\Omega \mathbb{V}^t\cdot\mathbb{W}\,\mathrm{d}x\right]; \qquad [74]$$

$\mathbb{D}$ depends on the geometry of the body and on the initial field.

We give now a physical interpretation of the additional term in $\mathbb{D}$. The form of [74] suggests that the tensor

$$\mathbb{I}_F(\mathbb{W}) = \rho\int_\Omega \mathbb{V}^t\cdot\mathbb{W}\,\mathrm{d}x \qquad [75]$$

plays a role of an inertia tensor of the fluid with respect to the given initial velocity $\mathbb{W}$. It is however, in general, not symmetric since no restrictions are imposed on $\mathbb{W}$ except for [62]. In fact, the notion of an inertia tensor of the fluid as a constant relating the boundary conditions to the kinetic energy of the flow has no sense in the context of the non-stationary motion since this relation is time dependent. Yet, it does have sense in the context of the quasi-stationary motion, where the velocity field retains the form of the steady Stokes field. In fact, if the tensor $\mathbb{I}_F$ is added to the particle's inertia tensor $\mathbb{I}_p$ to form a fictitious particle–fluid inertia tensor, the corresponding quasi-stationary motion

$$[\mathbb{I}_p + \mathbb{I}_F] \cdot \dot{\mathscr{U}} = -\mathbb{R} \cdot \mathscr{U} \qquad [76]$$

results in the same displacement matrix [74]. $\mathbb{I}_F$ may therefore be understood as an inertia tensor in a time-averaged sense. If the initial velocity field is taken to be the steady Stokes flow $\mathbb{V}$, [76] defines the quasi-stationary motion for which the velocity field at any moment is the steady Stokes field. $\mathbb{I}_F(\mathbb{V})$ is infinite due to the translation part, but the rotation part ${}^R\mathbb{I}_F({}^R\mathbb{V})$ may be finite for some symmetric geometries. In these cases it is symmetric, as may be seen from its definition, and the quasi-stationary and the non-stationary approaches lead to the same displacement matrix:

$$^R\mathbb{I}_F(^R\mathbb{V}) = \rho \int_\Omega {}^R\mathbb{V}^t \cdot {}^R\mathbb{V} \, dx. \qquad [77]$$

For the rotating sphere it is finite and diagonal. The diagonal elements are twice the kinetic energy E of the steady Stokes flow caused by a sphere rotating in an angular velocity of magnitude one. The total rotation is

$$\phi^\infty = -\Omega^{-1} \cdot [\mathbb{I}_r + 2E\mathbb{I}] \cdot \omega^0. \qquad [78]$$

It is identical with [58].

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